SP Numbers Over Integer Layers

Raghavendra Bhat, Cristian Cobeli and Alexandru Zaharescu

Department of Mathematics, University of Illinois at Urbana-Champaign

ABSTRACT

The dynamical system generated by the iterated calculation of the high order gaps between neighboring terms of a sequence of natural numbers is remarkable and only incidentally characterized at the boundary by the notable Proth-Glibreath Conjecture for prime numbers. We introduce a natural extension of the original triangular arrangement, obtaining a growing hexagonal covering of the plane. We focus on the sequence of Square-Primes (products of squares and primes) and derive some preliminary results on these numbers. This is joint work with Alexandru Zaharescu and Cristian Cobeli.

Background

- Let $\mathfrak{u} = \{a_k\}_{k\geq 0}$ be a sequence of non-negative integers.
- Place the sequence on the top row of a triangle.
- Obtain subsequent rows as sequences of numbers given by the absolute values of the differences between neighboring terms on the previous line.

$$
\begin{array}{cccccccccccc}
a_0 & & a_1 & & a_2 & & a_3 & & a_4 & & a_5 & \dots \\
& d_0^{(1)} & & d_1^{(1)} & & d_2^{(1)} & & d_3^{(1)} & & d_4^{(1)} & & \dots \\
& & d_0^{(2)} & & d_1^{(2)} & & d_2^{(2)} & & d_3^{(2)} & & \dots & \\
& & & d_0^{(3)} & & d_1^{(3)} & & d_2^{(3)} & & \dots & & \\
& & & & \dots & & \dots & & \dots & & &
\end{array}
\qquad \text{(P-G)}
$$

Conjecture (Proth-Gilbreath)

All the differences on the western edge of the (P-G) triangle generated by the sequence of all primes are equal to 1.

So far, it is not even known if there are an infinite number of 1's. We examine operator Υ that transforms the top sequence $\mathfrak{u}$ into the one on the left edge $\mathfrak{w}$ in the (P-G) triangle.

Υ is defined by

$$\Upsilon : \mathcal{L} \to \mathcal{L} \text{ and } \Upsilon(\mathfrak{u}) := \mathfrak{w}.$$

- Let $\Psi : \mathcal{L} \to \mathcal{L}$ be the operator that transforms a horizontal row of the triangle into the immediately following row.
- The entire triangle (P-G) is composed of the sequence of successive horizontal rows $\Psi^{(j)}(\mathfrak{u})$, for $j \geq 0$, where $\Psi^{(0)}(\mathfrak{u}) = \mathfrak{u}$ is the top generating row.
- The restrictions of Ψ and Υ to $\mathcal{L}_0$ have in their image only sequences in $\mathcal{L}_0$. Ex: $\mathbb{F}_2$

- The same type of property occurs with the action of Ψ and Υ on sequences $\mathfrak{s}$ of 0's and 1's, except for a finite number of terms.
- For these sequences, $\Psi(\mathfrak{s})$ is also ultimately composed only of 0's and 1's, but this is not generally true for $\Upsilon(\mathfrak{s})$.
- The geometric correspondent of each iteration of Υ applied on $\mathfrak{u}$ results in the construction of a new equilateral triangle, rotated clockwise around the first component a_0 of $\mathfrak{u}$ by an angle of 60 degrees.

- After six iterations of Υ, the initial sequence $\mathfrak{u}$ is geometrically reached again.
- This results in the completion of the first *layer* or *level* of what can further be seen as a *helicoidal discrete surface*, since in general $\Upsilon^{(6)}(\mathfrak{u}) \neq \mathfrak{u}$.
- Continuing the iterations produces a discrete *helicoid* denoted $\mathcal{H} = \mathcal{H}(\mathfrak{u})$, a "Riemann surface"-like structure of non-negative integers. Let $\mathcal{H}_n = \mathcal{H}_n(\mathfrak{u})$, $n \geq 1$, denote the n-th levels of the helicoid, so that

$$\mathcal{H} = \bigcup_{n\geq 1} \mathcal{H}_n. \qquad (1)$$

FILTERED RAYS OVER ITERATED ABSOLUTE DIFFERENCES ON LAYERS OF INTEGERS

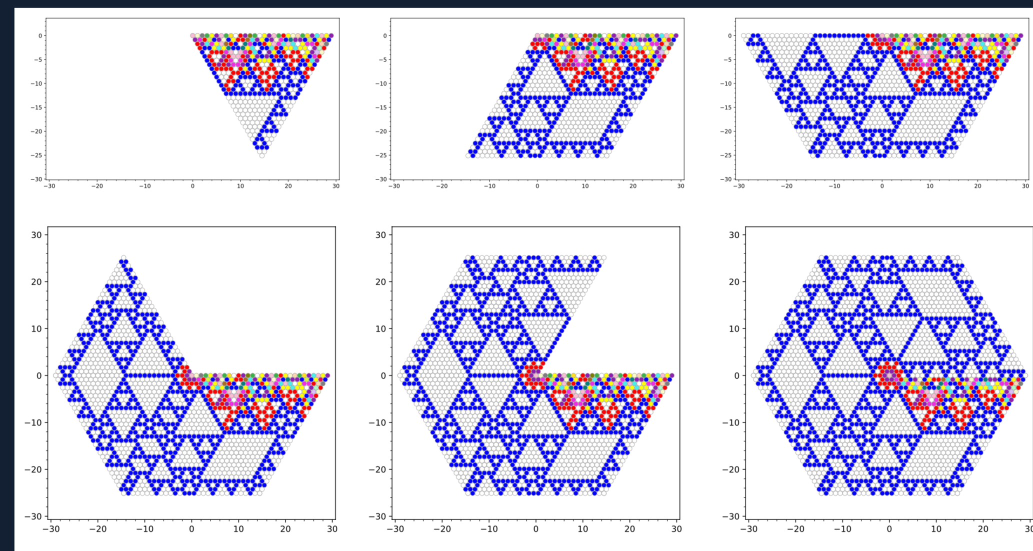

FIGURE 1. *The triangle generated by $\mathfrak{u}$ that contains the first 30 square-prime numbers. (See Section 3 for the formal definition and some appealing properties that the sequence of square-primes has.) Then, with $\Upsilon^{(n)}(\mathfrak{u})$ as initial rows, five more triangles are generated. The six figures represent the intermediate steps in forming the first layer of the helicoid.*

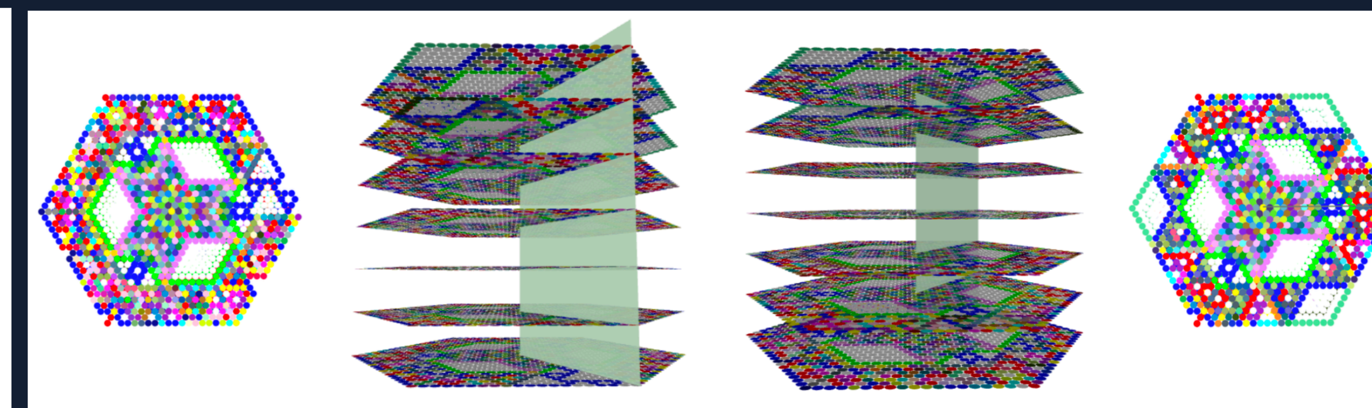

FIGURE 2. *The first seven layers of the helicoid generated by the sequence* $(100000, 59049, 32768, 16807, 7776, 3125, 1024, 243, 32, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0)$, *where the first positive integers are the first ten 5th powers in decreasing order. The four captures are taken in order from bottom, sides, and top. Distinct integers are shown in different colors. The helicoid has seven distinct layers, and starting from the 8th, all layers coincide with the 7th. The vertical strip indicates the places where the last outcome row on one layer transcends, becoming the first generating row on the upper layer.*

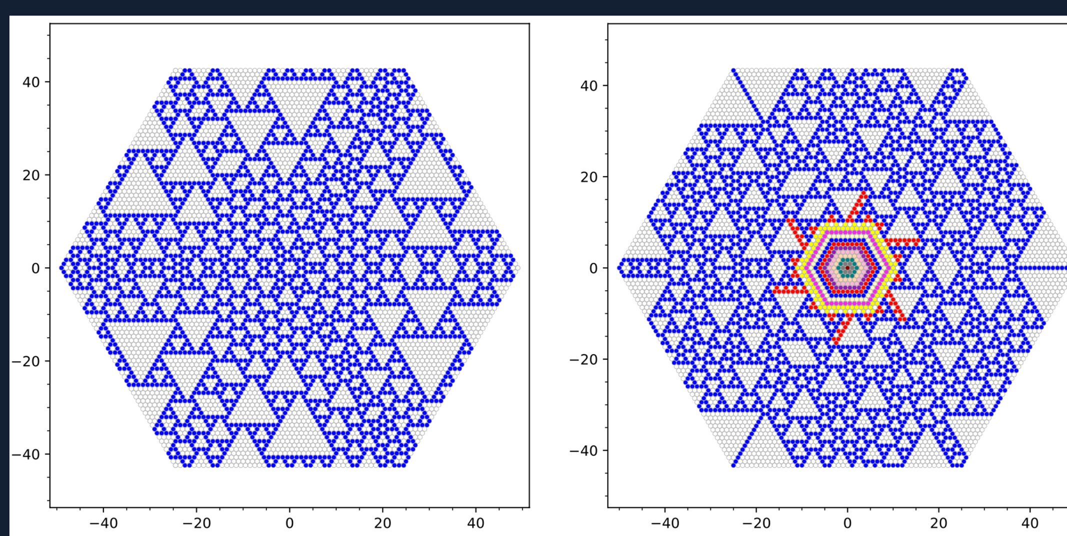

FIGURE 3. *Left: The base layer of the helicoid generated by the prime-number-indicator function for integers in the interval* $[0, 50)$. *Right: The base layer of the helicoid generated by the sequence:* $3072, 1536, 768, 384, 192, 96, 48, 24, 12, 6, 3$ *(the first 10 integers that are 3 times a power of 2 in decreasing order), followed by a sequence or 'random bits' defined by* $\mathrm{ind}_{[0,1/2)}\left(\{k\sqrt{2}\}\right)$ *for* $k = 1, 2, \dots, 40$. *(Note that by Theorem 4, in both cases, the helicoids have identical layers at each level.)*

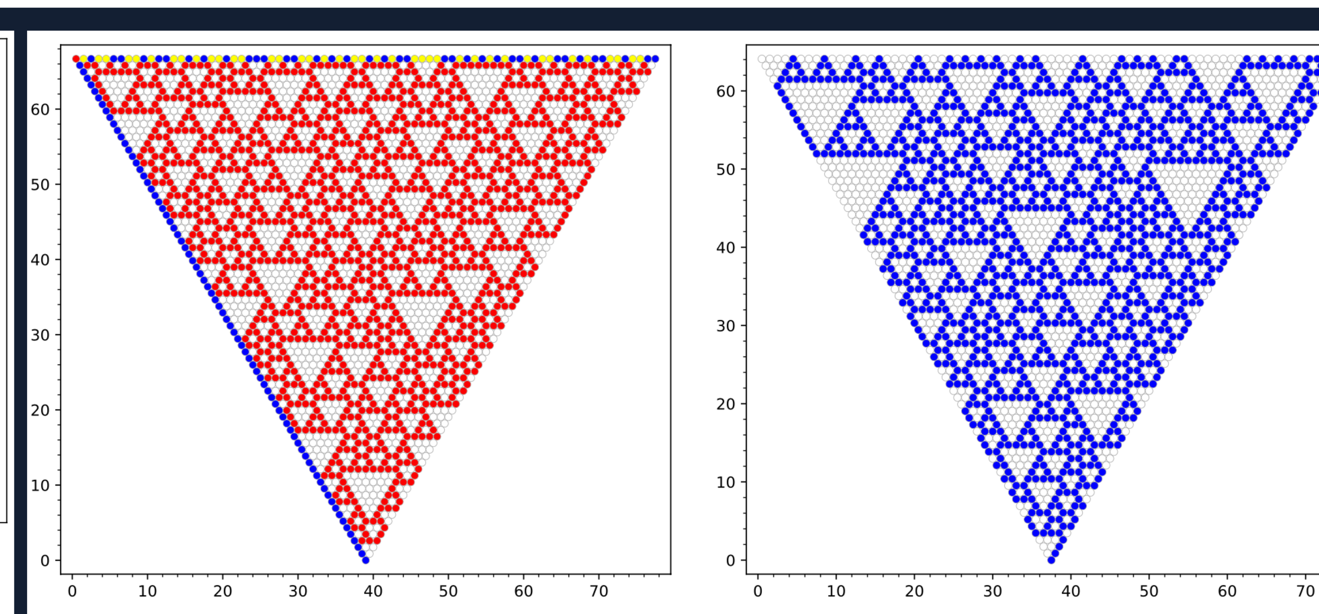

FIGURE 4. *Two cut-offs of the (P-G) triangles with integers taken modulo 4 (left) and modulo 2 (right). In the image on the left, the triangle is generated by the first 78 primes less than 400, and in the image on the right, the top row contains the 75 square-prime numbers less than 400.*

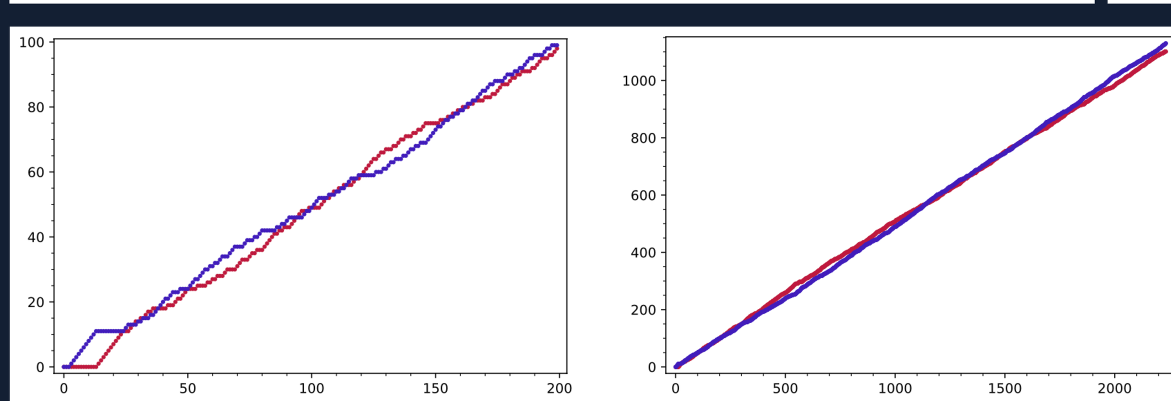

Figure 5: *The number of 1's versus the number of 0's on the left-edge of the (P-G) with square-prime numbers on the first row. The image on the left shows the first 200 values and the one on the right shows 2234 values obtained from the square-primes less than 20000. In total there are 1101 ones and 1130 zeros.*

Table 1: *The frequencies of the absolute values of the differences on the rays that cross a cut-off of the (P-G) triangle passing parallel to its left edge. The generating row contains the first 50 000 prime numbers:* $2, 3, \dots, 611\,953$. *All differences are reduced modulo 4. The notations are as follows: r is the number of the ray, starting with $r = 1$, the ray next to the left edge; N is the number of differences on the ray (note that there are no differences on the first row of (P-G)); z is the number of zeros and t is the number of two's.*

r	N	z	t	$(z-t)/N$
1	49998	24914	25084	-0.00340
2	49997	25095	24902	0.00386
3	49996	25033	24963	0.00140
4	49995	25019	24976	0.00086
5	49994	25074	24920	0.00308

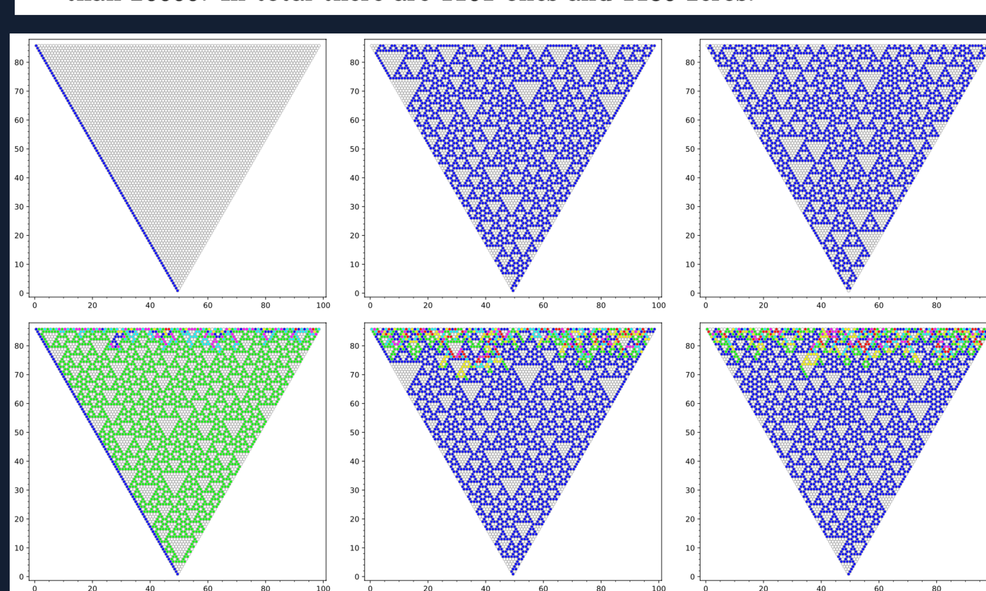

Figure 6: *The gaps in the (P-G) triangles generated by primes (left), square-primes (middle) and random numbers (right). The initial rows (not shown) contain the first one hundred primes, the first one hundred square-primes, and one hundred integers selected randomly from* $[2, 550]$, *respectively. (Note that* $p_{100} = 541$ *and* $s_{100} = 549$.) *The gaps are represented by two colors in the top triangles and by seven colors at the bottom. The colors correspond to the residue classes of the gaps* (mod 2) *and* (mod 7), *respectively. The triangles on the right side are obtained by two independent random choices of the numbers on the initial rows.*

TABLE 2. *The frequencies of the absolute values of the differences on the rays* $\mathfrak{w}_0, \dots, \mathfrak{w}_9$ *that cross a cut-off of the (P-G) triangle passing parallel to its left edge. The generating row contains the 69179 square-primes less than one million. Counting is done based on the values of the higher-order differences taken modulo 2. The notations are as follows: r is the rank of the ray, starting with the left edge $r_0 = \mathfrak{w}$; N is the number of differences on the ray (note that there are no differences on the first row of (P-G)); z is the number of 0's, o is the number of 1's, and h is the number of values that are not equal to 0 or 1.*

r	N	z	o	$z-o$	h	$(z-o)/N$
0	69178	34616	34559	57	3	0.00082
1	69177	34684	34485	199	8	0.00288
2	69176	34614	34556	58	6	0.00084
3	69175	34439	34727	-288	9	-0.00416
4	69174	34485	34681	-196	8	-0.00283
5	69173	34808	34357	451	8	0.00652
6	69172	34707	34458	249	7	0.00360
7	69171	34471	34694	-223	6	-0.00322
8	69170	34644	34522	122	4	0.00176
9	69169	34689	34472	217	8	0.00314

Results proved on SP Numbers

- The number of SP numbers smaller than a given number n is asymptotic to $\frac{n}{\log n}$.
- There are infinitely many SP twins (pairs of consecutive SP Numbers).
- For any positive integer x, there exist two SP numbers a, b such that $x = a - b$.
- Any positive integer appears infinitely often as a difference between square-primes.

Results proved in Paper

- There exists an infinite subsequence of square-primes $A_1 < A_2 < \dots$ such that the (P-G) triangle generated by $A_1, A_2, \dots$ on the first row has 1 as every other element on the left edge.
- Let $\mathfrak{u} = (a_0, a_1, \dots) \in \mathcal{L}_0$ be the first row of the (P-G) triangle and let $\mathfrak{w} = (b_0, b_1, \dots)$ be the sequence on its left-edge. Let $f, g \in \mathbb{F}_2[[X]]$ be the formal power series with coefficients in $\mathfrak{u}$ and $\mathfrak{w}$, respectively. Let $T : \mathbb{F}_2[[X]] \to \mathbb{F}_2[[X]]$ be the operator defined by $T(f) = g$. Then:
  1. The operator T satisfies the following formula
  $$\big(T(f)\big)(X) = f\left(\frac{X}{1+X}\right) \cdot \frac{1}{1+X}. \qquad (2)$$
  2. The operator T has the property $T^{(2)}(f) = f$ for any $f \in \mathbb{F}_2[[X]]$, so that T is invertible and bijective.
- Let $N \geq 1$ be an integer and let $\mathfrak{u} = (a_0, a_1, \dots, a_{N-1}) \in \mathcal{L}_0(N)$ be the top row and let $\mathfrak{w} = (b_0, b_1, \dots, b_{N-1})$ be the sequence on the left-edge of the (P-G) triangle of side N. Suppose $a_0 = b_0$ and let $f, g \in \mathbb{F}_2[X]^{[N-2]}$, be the polynomials whose coefficients are the components of $\mathfrak{u}$ and $\mathfrak{w}$, respectively. Let $T_N : \mathbb{F}_2[X]^{[N-2]} \to \mathbb{F}_2[X]^{[N-2]}$ defined by $T_N(f) = g$. Then:
  1. The operator T_N satisfies the following formula
  $$\big(T_N(f)\big)(X) \equiv f\left(\frac{X}{1+X}\right) \cdot \frac{1}{1+X} \left(\bmod X^N\right). \qquad (3)$$
  2. The operator T_N has the property $T_N^{(2)}(f) = f$ for any $f \in \mathbb{F}_2[X]^{[N-2]}$, so that T_N is invertible and bijective.
- Let $\boldsymbol{w} = \{w_j\}_{j\geq 1}$ be a sequence of non-negative integers. Then, there exists an increasing sequence of square-primes such that the (P-G) triangle they generate has on the western edge a sequence whose even-indexed elements are the elements of $\boldsymbol{w}$.

RELATED READING

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