Computational Experiments on Goldbach's Conjectures

Raghavendra N Bhat

University of Illinois, Urbana-Champaign
rnbhat2@illinois.edu

Fall 2020

1 Introduction

Famous mathematician Goldbach gave two conjectures. The first is the well known "every even number is a sum of two primes in at least one way". He also had another lesser known conjecture "every odd number is of the form $p + 2a^2$ where p is a prime and $a \in N$". This has two known contradictions namely 5777 and 5993.

2 Some notes on Goldbach's lesser known conjecture

Laurent Hodges(1) carried out computational experiments on solutions to the conjecture. We now wish to come up with more solutions to the problem using more powerful programs and computers.
Define a sequence s_n in the following way : the nth term of the sequence is the smallest odd number that can be represented as the sum

of a prime and twice a square in at least n ways.

We get the following sequence (first 50 terms) : 3, 5, 13, 31, 55, 61, 139, 139, 181, 181, 391, 439, 559, 619, 619, 829, 859, 1069, 1081, 1489, 1489, 1609, 1741, 1951, 2029, 2341, 2341, 3331, 3331, 3331, 3961, 4189, 4189, 4261, 4801, 4801, 5911, 5911, 5911, 6319, 6319, 6319, 8251, 8251, 8251, 8251, 8251, 8251, 9751, 9751, ...

(Note : There is a slight correction from the original literature as Hodges prints '3' twice. The smallest odd number with at least 2 solutions is 5 (0+5, 2+3).
Computationally, we see that 42871, 211069 and 410749 are the first odd numbers that can be expressed in at least 100, 200 and 300 ways respectively (thus, $s_{100} = 42871$, $s_{200} = 211069$ and $s_{300} = 410749$).
With computations spanning over 2 months, we get 1437679 as the first number with at least 500 unique solutions to Goldbach's lesser known conjecture. 5583451 is the first number with at least 900 unique solutions.

3 Conjectures

Conjecture 1 - On existence of primes on either side of x:

The popular Goldbach conjecture (on even numbers as sum of two primes) is equivalent to the following claim: $\forall\ x \in N$, $\exists\ m \in N$ such that $x + m$ and $x - m$ are both primes. Note that x need not be an even number. It can potentially be any natural number. The existence of primes on either side of it implies that $2x$ will be a sum of those two primes.

Let us define a function F(x) such that $\forall\ x \in N$, $F(x) = m$, where m is the smallest number such that $x + m$ and $x - m$ are both primes. Note that with each m we get a unique way of representing $2x$ as a

sum of two primes. Our function F(x) will only deal with the smallest such m. Eg : F(6) = 1, F(15) = 2.
Thus, F(x) indirectly gives us the solution to $2x$ as a sum of two primes which has the primes closest to each other.
Suppose the Goldbach conjecture were to be false, there would exist at least one x such that F(x) would not exist. In that case, $2x$ would fail to be a sum of two primes in at least one way.

Let us now define G(x) such that $\forall x \in N$, G(x)=max$\{F(n) | n \leqslant x\}$.
Thus, G(x) $\leqslant G(y)$ if $x \leqslant y$.

Conjecture : $\forall x \geqslant 8$, G(x)=$3k$ for some $k \in N$.

Thus, the peak values of F(x) turn out to be multiples of 3. Here is a graph of the function F(x) for the first 500 natural numbers.

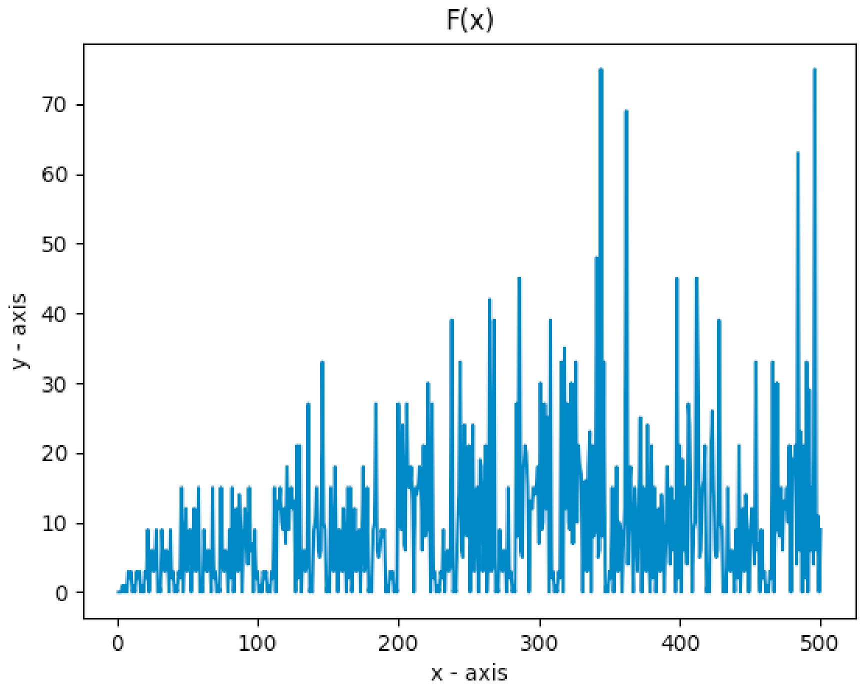

Figure 1: Claim is that peaks are always multiples of 3.

For the peaks of F(x) to be multiples of 3, it would mean that the x coordinates of the peaks have to be non-multiples of 3. This is because, suppose x itself was some multiple of 3, the m which would have to be added or subtracted from it to arrive at primes cannot be a multiple of 3. Since the function F(x) deals with values of m, only one of F(x) or x can afford to be a multiple of 3.

Now looking at G(x) for first 500 numbers. Note that G(x) is monotone and increasing as it takes the max value of F(n), for all n smaller than x.

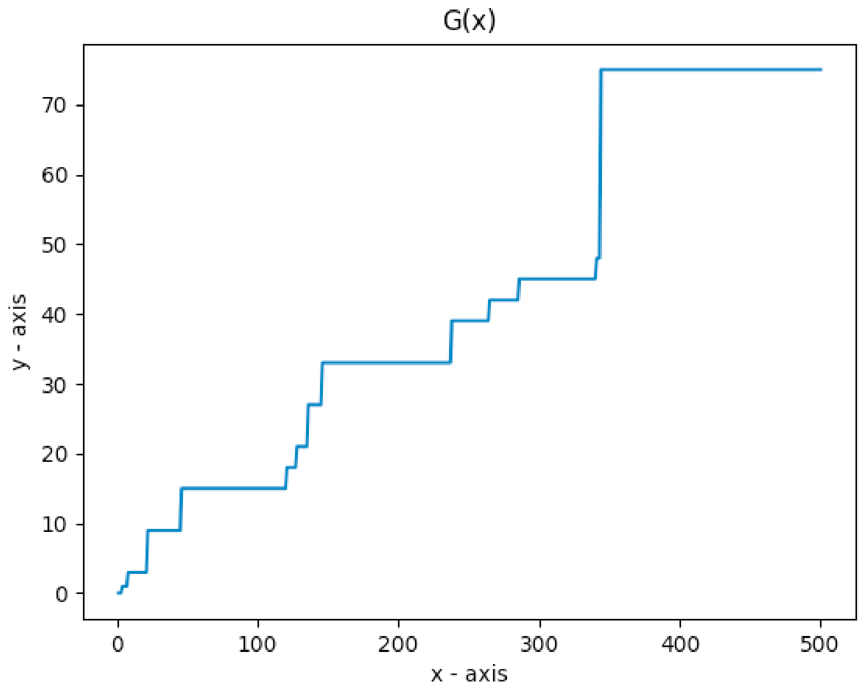

Figure 2: Claim is that values of G(x) are always multiples of 3 (beyond x=8)

These claims have been verified beyond tens of millions and there is no exception (for x greater than 8).

4 On Number of solutions to Goldbach

Let us now look at the number of solutions to the well known Goldbach conjecture, i.e, 'Every even number is a sum of two primes in at least one way'. For example, 4 has only one solution : 2+2. 10 has 2 solutions : 5+5 and 7+3. We count only unique solutions. Thus 3+7 and 7+3 are considered the same.

Let us define a 'peak' to be an even number that has more unique Goldbach solutions than any smaller even number.
4 is the first peak (1 solution : 2+2), 10 has 2 solutions (5+5, 3+7), 22 has 3 solutions (3+19, 5+17, 11+11) and so on. Thus, we get the following increasing sequence of peaks:

4, 10, 22, 34, 48, 60, 78, 84, 90, 114, 120, 168, 180, 210, 300, 330, 390, 420, 510, 630, 780, 840, 990, 1050, 1140, 1260, 1470, 1650, 1680, 1890...

(Note: The nth term need not have n Goldbach solutions. The sequence lists the peaks in ascending order. For example 114 has 10 Goldbach solutions, but the next peak 120 has 12 Goldbach solutions.)

Conjecture : Starting from 180, every peak is a multiple of 30. This has been verified beyond a million.
On Page 6 is a list of all peaks starting from 1000 up to 25,000 along with their respective number of unique Goldbach solutions.

A new way of defining Goldbach peaks:

While most part of this paper dealt with number of solutions to the Goldbach conjecture and also on minimizing the distance between the primes involved in the solutions (F(x), G(x) in section 3), one can

1050	57
1140	58
1260	68
1470	73
1650	76
1680	83
1890	91
2100	97
2310	114
2730	128
3150	138
3570	154
3990	163
4200	165
4410	171
4620	190
5250	198
5460	218
6090	222
6510	241
6930	268
7980	274
8190	292
9030	303
9240	329
10290	330
10710	340
10920	362
11550	393
13020	394
13650	433
13860	446
15330	447
15540	466
15960	477
16170	517
17850	530
18480	571
20790	615
21840	635
23100	671
24570	690
25410	719

Figure 3: List of numbers who are Goldbach solutions peaks

also define Goldbach solutions by maximizing the distance between the primes involved.

For a given even number, let us pick two primes that add up to the even number such that they are as far apart as possible. In informal terms, let us look at its largest Goldbach solution. Rather than defining a peak based on the distance between the two primes, let us define a peak using the product of the two primes.

In other words, $\forall$ x such that x is even, let H(x) = $p_i * p_j$ where p_i - p_j is the greatest among all pairs of primes that add up to form x. For example, H(8) = 15 i.e., 5*3.
Here is a list of the first few peaks in terms of products of primes (along with the actual product), for numbers up to 210 :

```
8 15 (3, 5)
10 21 (3, 7)
12 35 (5, 7)
16 39 (3, 13)
18 65 (5, 13)
24 95 (5, 19)
28 115 (5, 23)
30 161 (7, 23)
38 217 (7, 31)
52 235 (5, 47)
54 329 (7, 47)
60 371 (7, 53)
68 427 (7, 61)
80 511 (7, 73)
90 581 (7, 83)
96 623 (7, 89)
98 1501 (19, 79)
128 2071 (19, 109)
208 2167 (11, 197)
210 2189 (11, 199)
```

Figure 4: List of peaks up to 210

On the next page are the first few peaks larger than 210:

```
212 2587 (13, 199)
220 4531 (23, 197)
302 5377 (19, 283)
308 8587 (31, 277)
346 9193 (29, 317)
488 14167 (31, 457)
556 23923 (47, 509)
854 25513 (31, 823)
908 27187 (31, 877)
962 39517 (43, 919)
992 67087 (73, 919)
1382 80581 (61, 1321)
1768 80887 (47, 1721)
1856 119863 (67, 1789)
2078 123037 (61, 2017)
2438 144997 (61, 2377)
2618 155977 (61, 2557)
2642 261517 (103, 2539)
3818 295381 (79, 3739)
3848 297751 (79, 3769)
4618 322837 (71, 4547)
4886 351349 (73, 4813)
5348 353827 (67, 5281)
5372 727387 (139, 5233)
6008 746887 (127, 5881)
7426 1254769 (173, 7253)
9596 1481923 (157, 9439)
11642 1598917 (139, 11503)
12886 2199349 (173, 12713)
16502 2274457 (139, 16363)
18908 2832307 (151, 18757)
21368 3330127 (157, 21211)
22832 3695047 (163, 22669)
23456 3796759 (163, 23293)
30518 3859657 (127, 30391)
31478 4730377 (151, 31327)
34096 5868679 (173, 33923)
43376 6009943 (139, 43237)
43532 9140731 (211, 43321)
48002 9227137 (193, 47809)
54244 12584563 (233, 54011)
63274 18453433 (293, 62981)
84116 23724739 (283, 83833)
100768 25229767 (251, 100517)
111722 25531897 (229, 111493)
113672 35481367 (313, 113359)
128168 42314047 (331, 127837)
180596 55348723 (307, 180289)
```

Figure 5: List of peaks beyond 210

Conjecture : Every peak larger than 210 is a non-multiple of 3. This has been verified beyond a hundred million (the number itself being over 200 million. The product is beyond 100 Billion). For a peak to be a non-multiple of 3, it would be mean that the product itself is also a non-multiple of 3.

5 Codes

Here I give the python codes to run the functions F(x) and G(x) as discussed in this paper.

1. Code to evaluate F(x) as defined in section 3.

```
import math
def primetest(number):
    if number == 2:
       return True
    if number % 2 == 0:
        return False

    i = 3
    sqrtOfNumber = math.sqrt(number)

    while i <= sqrtOfNumber:
        if number % i == 0:
            return False
        i = i+2

    return True

def f(x):
    y=0
    if x==1:
        return y
    while True:
```

```
        if primetest(x+y)==True and primetest(x-y)==True:
            return y
        else:
            y+=1

x=2
while True:
    print(str(x)+ ' ' + str(f(x)))
    x+=1
```

2. Code to evaluate G(x) as defined in section 3.

```
import math
def primetest(number):
    if number == 2:
        return True
    if number % 2 == 0:
        return False

    i = 3
    sqrtOfNumber = math.sqrt(number)

    while i <= sqrtOfNumber:
        if number % i == 0:
            return False
        i = i+2

    return True
maxi=0
x=10
y=0
mmax=0
while True:
    if primetest(x+y)==True and primetest(x-y)==True:
        if y>maxi:
            print(str(x)+ ' ' + str(y))
```

```
            str(primetest((y*2)-1)))
            maxi=y
        x+=1
        y=0
        haha=1
    else:
        y+=1
```

6 References

[1] Hodges, Laurent. ‘A lesser known Goldbach conjecture’. Mathematics Magazine , Feb., 1993, Vol. 66, No. 1 (Feb., 1993), pp. 45-47.