# ON THE S'(K) FUNCTION

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## 1. Introduction

In this paper, we explore a summation function on natural numbers, namely $S'(k)$, defined as follows:

$$S'(k) = \sum_{h=1}^{2k-1} \sum_{j=1}^{2k-1} (-1)^{j+1+[hj/k]}$$

Here, $[hj/k]$ denotes the greatest integer lesser than $hj/k$. For simplicity of notation, we also define inner sums:

$$S'(k,h) = \sum_{j=1}^{2k-1} (-1)^{j+1+[hj/k]}$$

$$S'(k,h,j) = (-1)^{j+1+[hj/k]}$$

We prove some results, giving simpler formulas to compute $S'(k)$ for any odd $k$ and then make some conjectures about even $k$. Finally, we explore a smaller summation:

$$S(k) = \sum_{h=1}^{k-1} \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]}$$

Below are the first 20 values of $(k, S'(k))$:

$(1,-1)$,$(2,-1)$, $(3,-5)$, $(4,-1)$, $(5,-9)$, $(6,-9)$, $(7,-13)$, $(8,-1)$, $(9,-25)$, $(10,-17)$, $(11,-21)$, $(12,-17)$, $(13,-25)$, $(14,-25)$, $(15,-61)$, $(16,-1)$, $(17,-33)$, $(18,-49)$, $(19,-37)$, $(20,-33)$.

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## 2. Results on S'(k), k odd

We provide a general formula for $S'(k)$, when $k$ odd, that involves a single summation involving $h$, as opposed to the original double sum involving $h$ and $j$. Moreover, our formula does not involve sums over powers of $-1$.

**Theorem 1.**

$$S'(k) = 2k - 1 - \sum_{\substack{1 \le h \le 2k-1 \\ h \ odd}} 2 \cdot gcd(k,h), \ \ when\ k\ odd.$$

The above result can be further simplified using the well-known Euler Phi function, as follows:

$$S'(k) = 2k - 1 - 2\sum_{d|k} d \cdot \#\{h : gcd(k,h) = d : 1 \le h \le 2k-1 \text{ odd } \}$$

$$\Rightarrow S'(k) = 2k - 1 - 2\sum_{d|k} d \cdot \# \left\{ h : gcd\left(\frac{k}{d}, \frac{h}{d}\right) = 1 : 1 \le h \le 2k-1 \text{ odd } \right\}$$

Since we have $gcd(k,h) = gcd(k, h-k)$, we can rewrite the sum as

$$S'(k) = 2k - 1 - 2\sum_{d|k} d \cdot \# \left\{ m : gcd\left(\frac{k}{d}, m\right) = 1 : 1 \le m \le \left\lfloor \frac{2k-1}{d} \right\rfloor : m \text{ odd } \right\}$$

$$\Rightarrow S'(k) = 2k - 1 - 2\sum_{d|k} d \cdot \phi\left(\frac{k}{d}\right).$$

We now prove the theorem with the help of a few, useful lemmas.

**Lemma 1.1.** *$S'(k,h) = 1$ for $k$ odd and $h$ even.*

*Proof.* By definition,

$$S'(k,h) = \sum_{j=1}^{2k-1} (-1)^{j+1+[hj/k]} = \sum_{j=1}^{2k-1} S'(k,h,j)$$

For all $1 \le j < k$,

$$S'(k,h,j+k) = (-1)^{j+k+1+[h(j+k)/k]} = (-1)^{j+[hj/k]} = -S'(k,h,j).$$

We have

$$S'(k,h,k) = (-1)^{k+1+[hk/k]} = (-1)^{k+1+h} = 1.$$

Thus,

$$S'(k,h) = \sum_{j=1}^{k-1}(-1)^{j+1+[hj/k]} + \sum_{j=k+1}^{2k-1}(-1)^{j+1+[hj/k]} + S'(k,h,k) = 1.$$

□

**Lemma 1.2.** *For $k$ odd and $h$ odd, we have*

$$S'(k,h) = \sum_{\substack{1\le j\le 2k-1 \\ \frac{hj}{k}\in\mathbb{Z}}} (-1)$$

*Proof.* Consider any $j$ with $[hj/k] \neq hj/k$. Then,

$$\begin{aligned} S'(k,h,2k-j) &= (-1)^{2k-j+1+[h(2k-j)/k]} = (-1)^{2k-j+1+2h+[-hj/k]} \\ &= (-1)^{-j-[hj/k]} = (-1)^{j+[hj/k]} = -S'(k,h,j). \end{aligned}$$

Thus, within the summand of $S'(k,h)$, each $S'(k,h,j)$ cancels out $S'(k,h,2k-j)$, when $[hj/k] \neq hj/k$.

On the other hand, if $[hj/k] = hj/k$,

$$\begin{aligned} S'(k,h,2k-j) &= (-1)^{2k-j+1+[h(2k-j)/k]} = (-1)^{2k-j+1+2h+[-hj/k]} \\ &= (-1)^{-j+1-[hj/k]} = S'(k,h,j). \end{aligned}$$

Therefore, we conclude

$$S'(k,h) = \sum_{1\le h\le 2k-1} S'(k,h,j) = \sum_{\substack{1\le j\le 2k-1 \\ [hj/k]=hj/k}} (-1)^{j+1+hj/k}.$$

It is easy to see that $hj/k$ matches the parity of $j$ when $[hj/k] = hj/k$ since $h$ and $k$ both are odd. This gives

$$(-1)^{j+1+hj/k} = (-1).$$

Thus,

$$S'(k,h) = \sum_{\substack{1\le j\le 2k-1 \\ [hj/k]=hj/k}} (-1)^{j+1+hj/k} = \sum_{\substack{1\le j\le 2k-1 \\ \frac{hj}{k}\in\mathbb{Z}}} (-1).$$

□

**Lemma 1.3.** $S'(k,h) = 1 - 2\cdot gcd(k,h)$ *for $k$ odd and $h$ odd.*

*Proof.* From 1.2, we have that

$$S'(k,h) = \sum_{\substack{1\le j\le 2k-1 \\ \frac{hj}{k}\in\mathbb{Z}}} (-1).$$

Let $d = gcd(k,h)$. The least common multiple of $k$ and $h$ can be expressed as

$$lcm(k,h) = \frac{k\cdot h}{d}.$$

Thus, we have

$$\frac{hj}{k} \in \mathbb{Z} \Leftrightarrow j = \frac{n\cdot k}{d} \text{ for some } n \in \mathbb{N}.$$

Since $1 \le j \le 2k-1$, we have $1 \le n < 2d$, because $n = 2d$ gives $j = 2k$. Thus,

$$S'(k,h) = \sum_{1\le n<2d} S'(k,h,\frac{nk}{d}) = \sum_{1\le n<2d} (-1) = -(2d-1) = 1 - 2\cdot gcd(k,h).$$

□

*Proof. of Theorem 1:* From 1.1, 1.2, 1.3 we have:

$$S'(k) = \sum_{\substack{1\le h\le 2k-1 \\ h \text{ even}}} 1 + \sum_{\substack{1\le h\le 2k-1 \\ h \text{ odd}}} 1 - 2\cdot gcd(k,h), \text{ when } k \text{ odd}.$$

We combine the constants together to obtain:

$$S'(k) = 2k - 1 - \sum_{\substack{1\le h\le 2k-1 \\ h \text{ odd}}} 2\cdot gcd(k,h), \text{ when } k \text{ odd}.$$

This completes the proof of the theorem. □

As immediate consequences of the theorem, we get the following nice results:

**Corollary 1.1.** $S'(p) = 1 - 2p \ \forall p \in \mathbb{P}$.

*Proof.* From 1, we have

$$S'(p) = 2p - 1 - \sum_{\substack{1\le h\le 2p-1 \\ h \text{ odd}}} 2\cdot gcd(p,h).$$

Since $p$ is prime, $gcd(p,h) = 1$ for all $h$ not a multiple of $p$. In the interval of interest, there are $p$ odd numbers and the only multiple of $p$ is $p$ itself. Thus,

$$S'(p) = 2p - 1 - 2\cdot(gcd(p,p)) - 2\cdot(p-1) = 2p - 1 - 2p - 2p + 2 = 1 - 2p.$$

$\square$

**Corollary 1.2.** $S'(pq) = 4(p+q) - 6pq - 3$ *for any* $p, q \in \mathbb{P}$

*Proof.* From 1, we have

$$S'(pq) = 2pq - 1 - \sum_{\substack{1 \le h \le 2pq-1 \\ h \text{ odd}}} 2 \cdot gcd(pq, h).$$

We have $q$ odd multiples of $p$, each having a $gcd$ of $p$ with $pq$ and $p$ odd multiples of $q$, each having a $gcd$ of $q$ with $pq$, with $pq$ being the overlap. Thus, we have $p-1$ odd multiples of $q$, $q-1$ odd multiples of $p$ and $pq$ as the unique numbers that have a common factor greater than 1 with $pq$. In the interval of interest, there are $pq$ odd numbers. Thus,

$$S'(pq) = 2pq - 1 - 2(q-1)p - 2(p-1)q - 2pq - 2(pq - q + 1 - p + 1 - 1).$$
$$= 2pq - 1 - 2pq + 2p - 2pq + 2q - 2pq - 2pq + 2q - 2 + 2p - 2 + 2.$$
$$= 4(p+q) - 6pq - 3.$$

$\square$

## 3. Results on S'(k), k even

We will now give some results for $S'(k)$ when $k$ is even and claim some conjectures based on computations.

**Theorem 2.** $S'(2^n) = -1$ *for all natural numbers* $n$.

**Lemma 2.1.** $S'(k,k) = 1 - 2k$.

*Proof.* $S'(k,k) = \sum_{j=1}^{2k-1} (-1)^{2j+1} = -1 \cdot (2k-1) = 1 - 2k$. $\square$

*Proof. of Theorem 2:* We verify the statement of the theorem for $k = 1$ and $k = 2$ by brute force. We will prove the theorem for $k = 2^n$ with $n \ge 2$. Consider any $S'(k,h)$ with $h \ne k, 1 \le h \le 2k-1$ and $k = 2^n$ for some $n$. We will show that this is always 1.

$$S'(2^n, h) = \sum_{j=1}^{2^{n+1}-1} (-1)^{j+1+[hj/2^n]} = \sum_{j=1}^{2^{n+1}-1} (-1)^{j+1} \cdot (-1)^{[hj/2^n]}$$

Claim:

$$\sum_{j=1}^{2^{n+1}} (-1)^{j+1+[hj/2^n]} = 0$$

for any $h$.

Case 1: $h$ is odd. We claim that for $1 \leq j \leq k$,

$$(-1)^{j+1+[hj/k]} = -(-1)^{(j+k)+1+[h(j+k)/k]}.$$

*Proof:*

$$(-1)^{(j+k)+1+[h(j+k)/k]} = (-1)^{(j+k)+1+[hj/k+h]} = (-1)^{(j+k)+1+[hj/k]+h}$$

$$= -(-1)^{j+1+[hj/k]} \text{ since } h \text{ is odd and } k \text{ is even.}$$

This completes the proof for $h$ odd.

Case 2: $h$ is even. Clearly, following the arguments from case 1:

$$(-1)^{j+1+[hj/k]} = (-1)^{(j+k)+1+[h(j+k)/k]}.$$

Thus, our problem reduces to showing

$$\sum_{j=1}^{2^n} (-1)^{j+1+[hj/2^n]} = 0 \text{ i.e. } S(k,h) + S(k,h,k) = 0.$$

Recall that the single summation in $S(k,h)$ involves $j$ going from 1 to $2k-1$. Thus, we add the additional $S(k,h,k)$ to allow the summation to include $k = 2^n$. Since $h$ and $k$ are both even,

$$S(k,h,k) = (-1)^{k+1+[hk/k]} = (-1)^{k+1+h} = -1.$$

We should now prove that $S(k,h) = 1$ when $k = 2^n, h < k$, and $h$ even. Firstly, assume $h/2$ is odd.

$$S(k,h) = \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]} = \sum_{j=1}^{2^n-1} (-1)^{j+1+[hj/2^n]}$$

When $j = k/2$,

$$(-1)^{k/2+1+[h\cdot 2^n/2^{n+1}]} = (-1)^{k/2+1+h/2} = (-1)^{1+h/2} = 1.$$

Again, for $1 \leq j < k/2$ we have

$$S(k,h,j) = (-1)^{j+1+[hj/k]} = -(-1)^{j+k/2+1+[h(j+k/2)/k]} = -S(k,h,j+k/2).$$

Thus, $S(k,h) = 1$ as the $S(k,h,j)$ pairs around $j = k/2$ cancel out with the corresponding $S(k,h,j+k/2)$ pairs in the summation.

If $h/2$ is even, the same argument continues inductively. At $j = k/2$,

$$(-1)^{k/2+1+[h\cdot 2^n/2^{n+1}]} = -1.$$

On either side, the partial sums add up to 1, by induction, and are duplicates of one another. Within each partial sum, the same reduction continues inductively, until one arrives at an odd number at which point the 1 becomes the pivot, hence cancelling the terms around it. More clearly, if $h/2^m$ is odd, $j = k/2^m$ becomes a pivot. Thus, one can look at the final contribution as $1 + (-1) + 1 = 1$ where the two 1s come from the the left and right partial sums inductively.

Thus, $S(k,h) = 1 \Rightarrow \sum_{j=1}^{2^{n+1}} (-1)^{j+1+[hj/2^n]} = 1 + S(k,h,k) = 1 + (-1) = 0.$

At $j = 2^{n+1}$, we have $(-1)^{j+1+[hj/2^n]} = (-1)^{2^{n+1}+1+2h} = -1$. Thus,

$$S'(k,h) = \sum_{j=1}^{2^{n+1}-1} (-1)^{j+1+[hj/2^n]} = 1$$

We have a total of $2k-2$ choices for $h$, each contributing 1 to the $S'(k)$ sum. The $S'(k,k)$ contributes $1-2k$. Thus, for all $k$ that are powers of 2,

$$S'(k) = 2k - 2 + 1 - 2k = -1.$$

□

Computations show that the general formula for $S'(k)$, when $k$ even, does not differ much from 1.

**Conjecture 1.**

$$S'(k) = 2k - 1 - \sum_{\substack{1 \leq h \leq 2k-1 \\ h \equiv 2 \bmod 4}} 2 \cdot gcd(k,h), \ \textit{when } k \textit{ even.}$$

We are able to prove part of the result, with the following lemma.

**Lemma 1.1.** $S'(k,h) = 1$ *for* $k$ *even and* $h$ *odd.*

*Proof.*

$$S'(k,h) = \sum_{j=1}^{2k-1} (-1)^{j+1+[hj/k]} = \sum_{j=1}^{2k-1} S'(k,h,j)$$

For all $1 \le j < k$,

$$S'(k,h,j+k) = (-1)^{j+k+1+[h(j+k)/k]} = (-1)^{j+[hj/k]} = -S'(k,h,j).$$

We have

$$S'(k,h,k) = (-1)^{k+1+[hk/k]} = (-1)^{k+1+h} = 1.$$

Thus,

$$S'(k,h) = \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]} + \sum_{j=k+1}^{2k-1} (-1)^{j+1+[hj/k]} + S'(k,h,k) = 1.$$

□

## 4. Some notes on S(k)

We now provide some elementary results on a shorter sum,

$$S(k) = \sum_{h=1}^{k-1} \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]}.$$

**Theorem 3.** *$S(k)$ and $k$ have opposite parities.*

*Proof.* Recall two sub-functions

$$S(k,h) = \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]}$$

$$S(k,h,j) = (-1)^{j+1+[hj/k]}$$

In other words,

$$S(k) = \sum_{h=1}^{k-1} S(k,h) \text{ and } S(k,h) = \sum_{j=1}^{k-1} S(k,h,j).$$

Without loss of generality, assume $k$ to be odd. Thus, $k-1$ is even. For each $h$ between 0 and $k$, $S(k,h)$ involves a summation over $k-1$ different values of $S(k,h,j)$. Since each of these values is either $-1$ or 1, we have a summation over an *even* number of $-1$s and 1s. We conclude that $S(k,h)$ is even for all $h$ if $k$ is odd.

Thus, $S(k) = \sum_{h=1}^{k-1} S(k,h)$, which is essentially a summation over even numbers, is even.
The symmetrical argument holds true when $k$ is even, where we have each of the $S(k,h)$ equal an odd number. There are $k-1$ such pairs and we have the proof. □

**Corollary 3.1.** *(3,2) is the only tuple where both $k$ and $S(k)$ are prime.*

**Theorem 4.** *$S(k,h) = h$ when $h = k-1$.*

*Proof.* Recall that

$$S(k,h) = \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]} \text{ and } S(k,h,j) = (-1)^{j+1+[hj/k]}.$$

If $h = k-1$, for each value of $j$, we have that $[hj/k] = [(k-1)\cdot j/k] = j-1$.

Thus, $S(k,h,j) = (-1)^{j+1+j-1} = 1$ for all $h$. This completes the proof as there are a total of $k-1$ different values of $h$. □

**Theorem 5.** *If $k \equiv 1$ mod 4 and $k$ prime, then $S(k) \equiv 0$ mod 4.*

**Lemma 5.1.** *$S(k,h) \equiv S(k,k-h)$ mod 4 for all $1 \leq h \leq k-1$.*

*Proof.* We compare individual values of $S(k,h,j)$ and $S(k,k-h,j)$.

$S(k,h,j) = (-1)^{j+1+[hj/k]}$ and $S(k,k-h,j) = (-1)^{j+1+[(k-h)\cdot j/k]}$.
Since the $(-1)^{j+1}$ is a common factor for both, we just compare the $[hj/k], [(k-h)\cdot j/k]$ exponents.

**Case 1:** Suppose $j$ is odd. Then we have

$$(-1)^{[(k-h)\cdot j/k]} = (-1)^{[j-hj/k]} = (-1)^{[-hj/k]} \cdot (-1)^j = (-1)^{[-hj/k]} \cdot (-1)$$

**Case 1.1:** If $hj/k \in \mathbb{Z}$ we have

$$(-1)^{[-hj/k]} \cdot (-1) = (-1)^{[hj/k]^{(-1)}} \cdot (-1) = (-1)^{[hj/k]} \cdot (-1)$$

and hence not equal to $(-1)^{[hj/k]}$.

**Case 1.2:** If $hj/k \notin \mathbb{Z}$ we have
$$(-1)^{[-hj/k]} \cdot (-1) = (-1)^{(-1)\cdot[hj/k]-1} \cdot (-1) = (-1)^{(-1)\cdot[hj/k]}$$
and hence equal to $(-1)^{[hj/k]}$.

**Case 2:** Suppose $j$ is even. Then we have
$$(-1)^{[(k-h)\cdot j/k]} = (-1)^{[j-hj/k]} = (-1)^{[-hj/k]} \cdot (-1)^j = (-1)^{[-hj/k]}$$
**Case 2.1:** If $hj/k \in \mathbb{Z}$ we have
$$(-1)^{[-hj/k]} = (-1)^{[hj/k]^{(-1)}}$$
and hence equal to $(-1)^{[hj/k]}$.

**Case 2.2:** If $hj/k \notin \mathbb{Z}$ we have
$$(-1)^{[-hj/k]} = (-1)^{(-1)\cdot[hj/k]-1}$$
and hence not equal to $(-1)^{[hj/k]}$.

Note that since $k$ is prime, $lcm(k,h) = k \cdot h$. Thus, $hj/k \notin \mathbb{Z}\ \forall\ 1 \le h, j \le k-1$.

Thus we have
$$S(k,h,j) = S(k,k-h,j) \text{ when } j \text{ is odd.}$$
$$S(k,h,j) \neq S(k,k-h,j) \text{ when } j \text{ is even.}$$
However, $k \equiv 1 \bmod 4$. Thus, there are an *even* number of even values of $j$ between 1 and $k-1$. For each such $j$,
$$|S(k,h,j) - S(k,k-h,j)| = 2.$$
$$\text{Thus } \sum_{j \text{ even}} |S(k,h,j) - S(k,k-h,j)| \equiv 0 \bmod 4.$$
In conclusion, for any $h$
$$S(k,h) - S(k,k-h) = \sum_{1\le j\le k-1} S(k,h,j) - \sum_{1\le j\le k-1} S(k,k-h,j)$$
$$= \sum_{j \text{ odd}} S(k,h,j) - \sum_{j \text{ odd}} S(k,k-h,j) + \sum_{j \text{ even}} S(k,h,j) - \sum_{j \text{ even}} S(k,k-h,j)$$
$$\equiv 0 \mod 4$$

This completes the proof of the lemma. $\square$

*Proof. of Theorem 5.* Firstly, note that $S(k,h) = \sum_{j=1}^{k-1} S(k,h,j)$. Since $k-1$ is even and $S(k,h,j) \in \{1,-1\}$ for any $h, j$, we have that $S(k,h)$ is even for all $1 \leq h \leq k-1$.

By 5.1, we have that $S(k,h) + S(k,k-h) \equiv 0 \bmod 4$ for all $h$. Since there are a total of $\frac{k-1}{2}$ pairs of $\{S(k,h), S(k,k-h)\}$, we have

$$S(k) = \sum_{h=1}^{k-1} S(k,h) \equiv 0 \bmod 4.$$

$\square$

**Theorem 6.** *If* $k \equiv 3 \mod 4$ *and* $k$ *prime,* $S(k) \equiv 2 \mod 4$.

*Proof.* The proof is the same as 5, except that $S(k,h) \equiv S(k,k-h) + 2 \mod 4$ for all $h$. This is due to a slight change in Case 2.2 of 5.1, where we now have an *odd* number of even $j$ values.

$$\text{Thus } \sum_{j \text{ even}} |S(k,h,j) - S(k,k-h,j)| \equiv 2 \bmod 4.$$

Thus, $S(k) = \sum_{1 \leq h \leq \frac{k-1}{2}} S(k,h) + S(k,k-h) \equiv 2 \mod 4$, again since $k-1$ is odd. $\square$

**Conjecture 2.** $S(k)$ *values are distributed equally modulo 4.*

**Conjecture 3.** $S(p)$ *is always positive for* $p$ *prime.*

A partial result of this conjecture can be gotten from the following lemma, which is a consequence of 1.

**Lemma 3.1.** $S(p,h) = 0$ *for* $h$ *odd.*

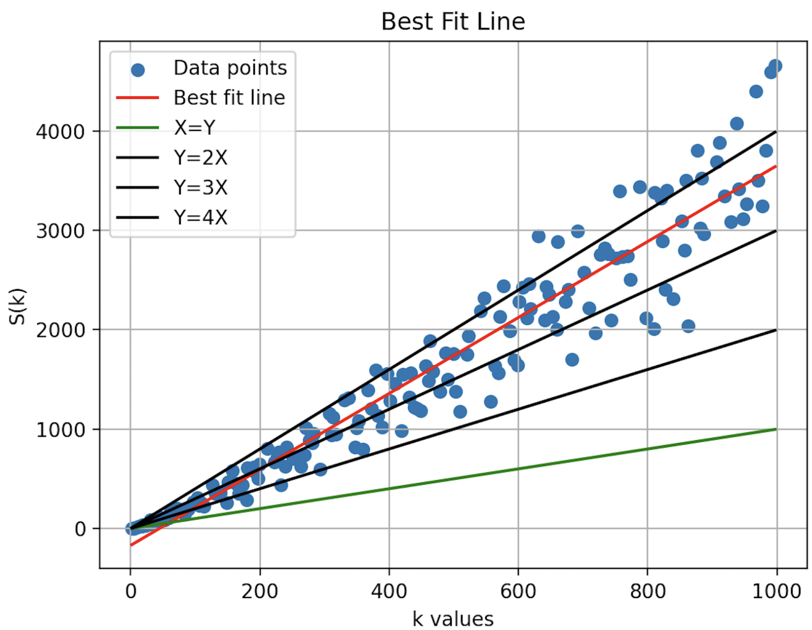
*Proof.* With a slight modification to the proof of 1.3, we get that $S(k,h) = 1 - gcd(k,h)$ when both $k, h$ are odd. Thus, $S(p,h) = 0$. $\square$

Thus, 3 reduces to showing that $\sum_{h \text{ even}} S(p,h) > 0$, where $2 \leq h \leq p-1$.

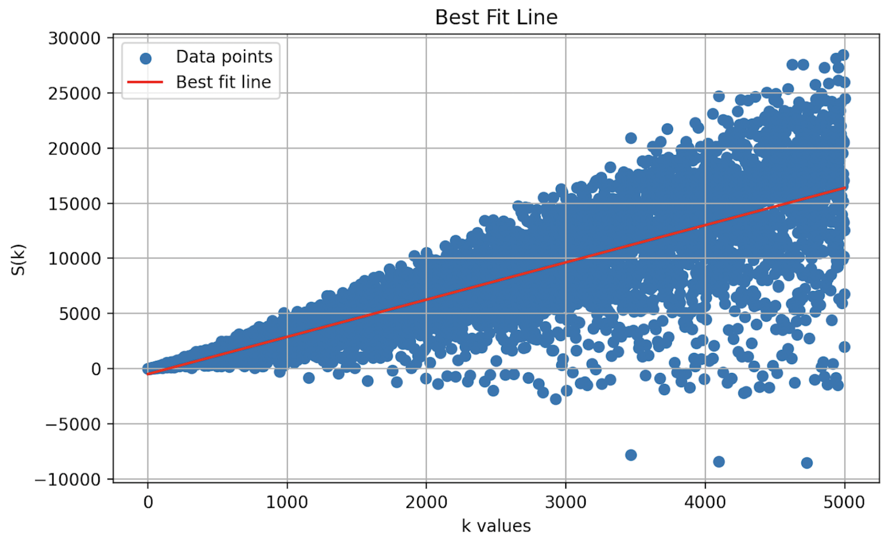
## 5. Some Illustrations

We now see a visual representation of 3. An ambitious claim would be the following:

**Conjecture 4.** *$S(p) > np$ for all $p$ prime, $n \in \mathbb{N}$, with $p$ sufficiently large, depending on $n$.*




Below is an image depicting the first 5000 values of $S(k)$. We see that there are quite a few negative values, as opposed to the values at primes only.

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